

Kink modes and effective four dimensional fermion and Higgs brane models

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Abstract

In the construction of a classical smoothed out brane world model in five dimensions, one uses a dynamically generated domain wall (a kink) to localise an effective four dimensional theory. At the level of the Euler-Lagrange equations, the kink sets up a potential well which can trap certain suitably coupled fields. This mechanism has been employed extensively to obtain localised, four dimensional, massless chiral fermions; these are crucial ingredients from the model building point of view. We present the generalisation of this kink trapping mechanism for both scalar and fermionic fields, and retain all degrees of freedom that were present in the higher dimensional theory. We show that a kink background induces a symmetric modified Pöschl-Teller potential well, and give explicit analytic forms for all the bound modes and a restricted set of the continuum modes. We demonstrate that it is possible to confine an effective four dimensional scalar field with a quartic potential of arbitrary shape. This can be used to place the standard model electroweak Higgs field on the brane, and also generate nested kink solutions. We also consider the limits of the parameters in the theory which give thin kinks and localised and de-localised scalar and fermionic fields.

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I. INTRODUCTION

Our physical universe is described extremely well by the standard model and general relativity, both of which are expressed in one time and three space dimensions. But these models are incomplete, and a large amount of work has been done addressing the question: can we improve our model of the universe by augmenting it with one or more extra spatial dimensions? Extra dimensions give extra degrees of freedom and these have been used to tackle a variety of problems present in the existing models.

There has been a vast investment of research into serious extra dimensional models; Universal Extra Dimensions [1], the Arkani-Hamed Dimopoulos and Dvali model [2, 3, 4] and the Randall-Sundrum model [5, 6] to name a few. The later demonstrated that large, even infinite, extra dimensions are not ruled out by experiment. In these large extra dimensional scenarios, one generically has the concept of a brane, to where the energy which we observe in our four dimensional world is concentrated, or confined. The majority of the Randall-Sundrum (RS) models assume the existence of a higher theory (e.g. string theory) which provides such a brane along with a confinement mechanism. The effective low energy description of this set-up consists simply of the combination of a five dimensional bulk action, and a four dimensional action placed by hand at a specific location in the extra dimension. The branes in these models are infinitely thin and Dirac delta distributions are used to couple the five and four dimensional sectors.

If our four dimensional universe is embedded in higher dimensions in such a way, then given enough energy, it should be possible to probe the structure of our brane. Theoretical predictions then require a more sophisticated description of the dynamics of the brane than just a delta distribution. Ideally, one would like to give a dynamical explanation for the formation of a brane, and the ensuing localisation, using classical field theoretic ideas. The canonical example of this comes from the work done initially by Rubakov and Shaposhnikov in [7] (see also [8, 9]). One has a five dimensional scalar field $\Phi(x^\mu, w)$ in a potential with \mathbb{Z}_2 symmetry that admits a topologically stable solution $\phi_c(w) \sim \tanh(mw)$. The mass parameter m controls the tension or inverse width of this domain wall defect. A massless five dimensional fermion with a Yukawa coupling to this kink has a Dirac equation which admits a separated solution $\Psi(x^\mu, w) \sim \psi_L(x^\mu) \cosh^{-a}(mw)$. The parameter a is proportional to the five dimensional Yukawa coupling constant and ψ_L is a massless left-handed four dimensional

fermion. In the limit of a thin brane, the extra dimensional factor of Ψ becomes a delta distribution, giving essentially the RS localisation mechanism, but with known dynamical origins.

Certainly, if one is just interested in the exact thin brane limit, then all degrees of freedom freeze out except for the massless zero mode ψ_L . But if the brane is dynamically generated, this limit will only be approximate, and at high enough energies the excited states (essentially the Kaluza-Klein modes of the infinite extra dimension) will become phenomenologically important. Gaining an understanding of the behaviour of these excitations is therefore a prerequisite for any realistic model building attempts.

Even with a full grasp of the dynamics of the confinement of massless chiral modes, one is still not in a position to dynamically embed the standard model in an extra dimension. In any realistic interacting model, there will be gauge symmetries that need to be broken, most notably the electroweak symmetry. The most straightforward idea is to localise the standard Higgs field to the brane, and then proceed in the usual manner of spontaneous symmetry breaking. Such a mechanism requires an analysis of the five dimensional couplings needed to produce an effective four dimensional quartic potential, and a compatible way of confining gauge fields.

In this paper we address some of these model building issues by demonstrating that it is possible to confine an effective Higgs field to a kink. We also present the full spectrum of modes of the kink, a coupled scalar field and a coupled fermion field in the canonical kink scenario. Along with this mode analysis, we also investigate the relevant limits of the parameters in the model which yield a thin kink, and show that we can obtain just what is needed for an embedding of the standard model. We do not address the issue of gauge field confinement; one promising idea is the Dvali-Shifman mechanism [10]. Gravity will also be ignored for the sake of simplicity in this initial pass through the problem.

We begin in Section II by presenting the toy model which supports a scalar kink and determine the full spectrum of its associated modes. We discuss the different limits of this model which give the scenarios with no kink, a thick kink and a thin kink. In Section III we add a scalar field to the kink model, and show that the kink sets up a symmetric modified Pöschl-Teller potential for the extra dimensional component of the scalar field. We determine the modes of this potential and use them to obtain an effective four dimensional action, discussing in detail the thin kink limit. In Section IV we analyse a fermion coupled

to the kink, present the full mode decomposition, and show that in the thin kink limit, the massless left-handed mode is the only surviving dynamical field. We also present an action that contains this massless four dimensional mode coupled to a five dimensional field. We conclude and discuss further work in Section V. Appendix A contains analytic solutions of the potential well set up by the kink which are used extensively throughout the analysis.

II. THE KINK AND ITS LIMITS

All subsequent work will be in $4 + 1$ dimensional Minkowski space with metric $g_{MN} = \text{diag}(1, -1, -1, -1, -1)$, where capital Latin letters index (t, x, y, z, w) and Greek letters index the $3 + 1$ dimensional subspace (t, x, y, z) . We begin with an action describing a real five dimensional scalar field Φ , given by

$$\mathcal{S}_\Phi = \int d^5x \left[\frac{1}{2} \partial^M \Phi \partial_M \Phi - V(\Phi) \right] \quad (1)$$

$$\text{with } V(\Phi) = \frac{a}{4m} \left(\Phi^2 - \frac{m^3}{a} \right)^2, \quad (2)$$

where a is a dimensionless constant¹ and m is the mass of Φ . From this action we find the Euler-Lagrange equation for Φ to be

$$\partial^M \partial_M \Phi - m^2 \Phi + \frac{a}{m} \Phi^3 = 0. \quad (3)$$

With the aim of producing an effective four dimensional theory, Φ will initially be taken to depend only on the extra dimensional coordinate w ; this behaviour is denoted as $\phi_c(w)$. A topologically stable solution to equation (3) is then

$$\phi_c(w) = \sqrt{\frac{m^3}{a}} \tanh \left(\frac{mw}{\sqrt{2}} \right), \quad (4)$$

which is the classical kink solution interpolating between the \mathbb{Z}_2 degenerate minima of the potential V . The energy density for this kink in the $3 + 1$ dimensional subspace is given by

$$\varepsilon_{\phi_c} = \int_{-\infty}^{\infty} dw \left[\frac{1}{2} (\partial_w \phi_c)^2 + V(\phi_c) \right] = \frac{2\sqrt{2}m^4}{3a}. \quad (5)$$

¹ The constant a is the five dimensional analogue of λ in the four dimensional potential $V(\phi) = \frac{\lambda}{4} (\phi^2 - \frac{m^2}{\lambda})^2$.

We are interested in the behaviour of models where the classical background is a thin kink, meaning m is very large. To make this more precise, we write our parameters as

$$a = \tilde{a}\Lambda^\alpha, \quad m = \tilde{m}\Lambda^\mu,$$

with \tilde{a} and \tilde{m} finite, and consider the limit $\Lambda \rightarrow \infty$. In such a limit the kink energy density is $\varepsilon_{\phi_c} \sim \Lambda^{4\mu-\alpha}$ and must remain finite, giving the constraint $\alpha = 4\mu$. This in turn means the amplitude of the kink is $|\phi_c| \sim \Lambda^{-\frac{1}{2}\mu}$. The single parameter μ now describes all possible limiting scenarios of the theory, with $\mu = 0$ corresponding to no limit being taken. If $\mu < 0$ the potential V disappears, $\phi_c \rightarrow 0$ and the kink solution is replaced by a plane wave. The case $\mu > 0$ is the more interesting thin kink limit. Here, the width of ϕ_c tends to zero and to keep the energy density finite, the height also vanishes. We will refer extensively to these limits in the following sections.

By assuming that Φ depends only on w we have of course lost a lot of the dynamics of the full theory. First, since ϕ_c breaks translational invariance along w , we expect a zero mode which can act to translate the kink. Second, if we have a thick kink, we expect there to be massive modes associated with arbitrary deformations of the kink. We now proceed to incorporate these dynamics.

A. Kink modes and the effective model

The classical kink background ϕ_c breaks the five dimensional Poincaré symmetry, leaving a four dimensional Poincaré subgroup. This makes it natural to decompose a field into a sum of products of an extra dimensional component and a $3 + 1$ dimensional component. For Φ , we want this expansion to be made about the kink solution, and so we take

$$\Phi(x^\mu, w) = \phi_c(w) + \sum_i \phi_i(x^\mu) \eta_i(w), \quad (6)$$

where η_i are a fixed orthonormal basis of the extra dimension, ϕ_i are four dimensional dynamical fields and the sum over i can in general be a combination of discrete and continuous modes. We would like to determine a basis η_i such that the equations of motion for the ϕ_i describe massive scalar fields. This can be done in the standard way by taking the action for Φ given by (1), substituting the expansion (6), using the fact that ϕ_c satisfies (3), discarding terms $\mathcal{O}(\phi_i \eta_i)^3$ and higher and using integration by parts. The effective second order action

is then

$$\mathcal{S}_\Phi^{\mathcal{O}(2)} = \int d^5x \left[\frac{a}{4m} \phi_c^4 - \frac{m^5}{4a} - \frac{1}{2} \phi_i \eta_i \left(\partial^\mu \partial_\mu - \partial_w^2 + \frac{3a}{m} \phi_c^2 - m^2 \right) \phi_j \eta_j \right],$$

with implicit sum i, j over the modes. For the ϕ_i to satisfy the massive Klein-Gordon equation with mass λ_i we require

$$\left(-\frac{d^2}{dw^2} + \frac{3a}{m} \phi_c^2 - m^2 \right) \eta_i = \lambda_i^2 \eta_i.$$

We can use the known form of ϕ_c to get

$$\left(-\frac{d^2}{dz^2} + 6 \tanh^2 z - 2 \right) \eta_i = \left(\frac{2\lambda_i^2}{m^2} \right) \eta_i, \quad (7)$$

where $z = mw/\sqrt{2}$. This differential equation is a Schrödinger equation with a symmetric modified Pöschl-Teller potential. Analytic solutions are known in terms of hypergeometric functions, and in general there are both bound and continuum solutions. In Appendix A we present these solutions expressed in terms of regular functions, along with their normalisation coefficients, for a more general form of the potential. For the kink modes at hand, equation (7) is equation (A.1) with $l = 2$, and so there are two bound modes and a continuum

$$\begin{aligned} \lambda_0^2 = 0 & & \eta_0(w) &= E_0 \cosh^{-2} z, \\ \lambda_1^2 = \frac{3}{2} m^2 & & \eta_1(w) &= E_1 \sinh z \cosh^{-2} z, \\ \lambda_q^2 = \frac{1}{2} (q^2 + 4) m^2 & & \eta_q(w) &= E_q e^{iqz} (3 \tanh^2 z - (q^2 + 1) - 3iq \tanh z). \end{aligned}$$

The bound $\eta_{0,1}$ are square integrable normalised by (A.2) and the continuum η_q are delta function normalised by (A.3), the normalisation constants being

$$E_0 = \sqrt{\frac{3m}{4\sqrt{2}}}, \quad E_1 = \sqrt{\frac{3m}{2\sqrt{2}}}, \quad E_q = \sqrt{\frac{m}{2\pi\sqrt{2}(q^2 + 1)(q^2 + 4)}}.$$

Armed with the basis η_i , we return to the analysis of the full dynamics of the kink. Expanding the original action (1) with Φ decomposed in the η_i basis and integrating over the extra dimension gives

$$\mathcal{S}_\Phi = \int d^4x [-\varepsilon_{\phi_c} + \mathcal{L}_\phi], \quad (8)$$

where the ϕ kinetic, mass and self coupling terms are

$$\begin{aligned}\mathcal{L}_\phi = & \frac{1}{2}\partial^\mu\phi_0\partial_\mu\phi_0 + \frac{1}{2}\partial^\mu\phi_1\partial_\mu\phi_1 - \frac{3}{4}m^2\phi_1^2 \\ & + \int_{-\infty}^{\infty} dq \left[\frac{1}{2}\partial^\mu\phi_q^*\partial_\mu\phi_q - \frac{1}{4}(q^2 + 4)m^2\phi_q^*\phi_q \right] \\ & - \kappa_{ijk}^{(3)}\phi_i\phi_j\phi_k - \kappa_{ijkl}^{(4)}\phi_i\phi_j\phi_k\phi_l.\end{aligned}\tag{9}$$

The fields $\phi_{0,1}$ are the real valued scalars associated with the two bound state modes $\eta_{0,1}$. The integration over q is over the complex valued continuum modes ϕ_q associated with η_q . Note that $\phi_{-q} = \phi_q^*$ and $\eta_{-q} = \eta_q^*$ and so this integral is real. The effective cubic and quartic self interaction couplings are

$$\begin{aligned}\kappa_{ijk}^{(3)} &= \frac{a}{m} \int_{-\infty}^{\infty} dw [\phi_c \eta_i \eta_j \eta_k], \\ \kappa_{ijkl}^{(4)} &= \frac{a}{4m} \int_{-\infty}^{\infty} dw [\eta_i \eta_j \eta_k \eta_l].\end{aligned}$$

For brevity, the indices i, j, k, l label bound modes, continuum modes or a mixture of both and the sum over these labels is implied in equation (9). The couplings κ can be computed as their integrands are known; some are zero due to parity, some are non-zero.

Equations (8) and (9) are exact manipulations of the original five dimensional Φ model (1), and provide a description in a basis useful for investigating the effective four dimensional behaviour. The bound modes are reminiscent of the Kaluza-Klein modes one obtains in compact extra dimensions, but in the case at hand the gaps in the mass spectrum are not uniform. Furthermore, the continuum modes do not have analogues in the Kaluza-Klein model and are not strictly four dimensional, but instead form a pseudo-five dimensional field with reduced degrees of freedom.

Regarding the renormalisability of our model, we note that because the original five dimensional action (1) contains non-renormalisable terms, we do not have any reason to stop writing down potentials at quartic order. But the manipulations which bring us to the four dimensional level given by equation (9), do in fact leave us with a renormalisable theory if we truncate the action to just the bound states. This renormalisability of the bound states may be a useful criteria for restricting the types of terms that one begins with in the five dimensional (or higher dimensional) action.

B. Limiting behaviour of the effective model

Now that we have such a reformulation of the kink model, we are in a position to analyse the full dynamics of the system for the three different limits of the mass m . Following our previous parameterisation, for $\mu < 0$ there is no kink and the basis η_i is replaced by a standard complex Fourier expansion. All the dynamical components are packaged together by the Fourier transform and it is no longer sensible to perform the w integral. Instead one should consider Φ as a free, massless, five dimensional field.

For the thick kink case when $\mu = 0$, the spectrum consists of the energy density of the integrated kink, a zero mode, a massive bound mode and a continuum of massive complex scalars. These effective four dimensional fields are self coupled and coupled amongst each other via cubic and quartic interactions. In particular, the zero mode ϕ_0 and massive bound mode ϕ_1 each have a potential

$$V_0(\phi_0) = \frac{9\sqrt{2}a}{140}\phi_0^4,$$

$$V_1(\phi_1) = \frac{3}{2}m^2\phi_1^2 + \frac{3\pi}{32}\sqrt{\frac{3a}{2\sqrt{2}}}m\phi_1^3 + \frac{9\sqrt{2}a}{280}\phi_1^4,$$

which are due to the non-zero values of $\kappa_{0000}^{(4)}$, $\kappa_{111}^{(3)}$ and $\kappa_{1111}^{(4)}$. Similarly, we can compute the coupling potential amongst these bound modes

$$V_{0,1}(\phi_0, \phi_1) = \frac{9\pi}{64}\sqrt{\frac{3a}{2\sqrt{2}}}m\phi_0^2\phi_1 + \frac{9\sqrt{2}a}{70}\phi_0^2\phi_1^2.$$

Note that while ϕ_0 has no mass term it does have a non-zero potential, making it energetically unfavourable to excite the field. This means that it costs energy to translate the kink. We can account for this unexpected result by recalling that $\eta_0(w)$ corresponds to *infinitesimal* translations of $\phi_c(w)$. Adding any small but finite multiple of $\eta_0(w)$ to $\phi_c(w)$ will, to first order, perform a translation, but to higher order it will *deform* the kink. The energy cost of these higher order deformations are described by the potential V_0 . We speculate that in a quantised version of the theory, V_0 will induce radiative corrections to the self energy of ϕ_0 . This in turn may be important when calculating loop corrections to the mass of the kink soliton, as is done in [11]. It is out of the scope of this paper to pursue such a speculation.

For completeness we point out that while V_1 has a cubic term, the potential has only one extremum, which is a minimum at $\phi_1 = 0$. This implies that ϕ_1 will not pick up a

vacuum expectation value, which is as expected because we have expanded about the kink and already incorporated the background due to the vacuum.

We now move on to the thin kink limit where $\mu > 0$ and both m and a tend to infinity. The kink energy density remains finite, but the masses of the bound mode ϕ_1 and the continuum modes ϕ_q go like m . Recall that these massive modes correspond to deformations in the kink and as the kink gets thinner it also gets stiffer, requiring more energy for a given deformation. The dynamics of the massive modes are thus frozen out, as they are no longer able to deform the kink without possessing infinite energy. It seems natural then that we are left with the energy of the kink and its translation zero mode. But there is also the potential of this zero mode to consider, V_0 , which in the thin kink limit will blow up unless $\phi_0 = 0$. This can be understood from the arguments give above: for an infinitely stiff kink, the higher order deformations due to the zero mode are forbidden and the only physical resolution is to remove the dynamics of the zero mode. So in the thin kink limit, we are left with only the kink energy density ε_{ϕ_c} in the effective four dimensional action (see [12] for a discussion of the translational zero mode in the presence of gravity).

III. ADDING A SCALAR FIELD

We have so far performed an analysis of the kink and its modes in isolation. As stated previously, we aim to use the properties of the domain wall to dynamically trap five dimensional fields to a brane and create an effective four dimensional model. We can achieve this if the kink is coupled to a different five dimensional field and projects out a set of modes with the lowest mode separated from the rest by a significant mass gap. Then if we are at energies where only the lowest bound state can be excited, the degree of freedom of propagation along the extra dimension has been lost and the bound mode is confined.

A. Scalar modes

The simplest place to start is to take another five dimensional scalar field $\Xi(x^M)$ and couple it to the kink field. The action describing this model is $\mathcal{S}_{\Phi+\Xi} = \mathcal{S}_{\Phi} + \mathcal{S}_{\Xi}$ where

$$\mathcal{S}_{\Xi} = \int d^5x \left[\frac{1}{2} \partial^M \Xi \partial_M \Xi - \frac{ab(b+1)}{4m} \Phi^2 \Xi^2 - W(\Xi) \right] \quad (10)$$

$$\text{with } W(\Xi) = \frac{n^2}{2} \Xi^2 + \frac{c}{4n} \Xi^4. \quad (11)$$

The parameters b and c are dimensionless, while the dimensionful parameter n is the mass of Ξ . We follow the analysis for the kink modes and perform the general separation

$$\Xi(x^M) = \sum_i \xi_i(x^\mu) k_i(w),$$

where the sum over i can again be a combination of discrete and continuous parts. To obtain a suitable basis k_i , we look at the linearised equation of motion for Ξ with $\Phi = \phi_c$

$$\left(\partial^\mu \partial_\mu - \partial_w^2 + \frac{ab(b+1)}{2m} \phi_c^2 + n^2 \right) \xi_i k_i = 0.$$

We want ξ_i to satisfy the four dimensional Klein-Gordon equation with mass δ_i . This leads to

$$\left(-\frac{d^2}{dz^2} + b(b+1) \tanh^2 z \right) k_i = \left(\frac{2(\delta_i^2 - n^2)}{m^2} \right) k_i$$

where $z = mw/\sqrt{2}$ as before. This Schrödinger equation has the same form as the one obtained for the kink modes. We see that the kink sets up a symmetric modified Pöschl-Teller potential well which traps not only its own modes, but also those of a coupled scalar field. Looking to Appendix A we see that the basis k_i contains $[b]$ bound modes² and a continuum. The masses of the bound states are

$$\begin{aligned} \delta_0^2 &= n^2 + \frac{1}{2} b m^2, \\ \delta_1^2 &= n^2 + \frac{1}{2} (3b-1) m^2, \\ \delta_2^2 &= n^2 + \frac{1}{2} (5b-4) m^2, \\ &\vdots \\ \delta_i^2 &= n^2 + \frac{1}{2} ((2i+1)b - i^2) m^2, \end{aligned}$$

² We use the standard notation $[.]$ for the ceiling function.

and for the continuum we have

$$\delta_q^2 = n^2 + \frac{1}{2}(q^2 + b(b+1))m^2,$$

where $q \in \mathbb{R}$ labels the continuum modes. We will not give explicit forms of the functions k_i ; they are easily determined from Appendix A. Unlike the modes of the kink, this spectrum of masses does not in general include a zero mode and the bottom of the spectrum is dependent on the parameters n , b and m . We also have the freedom to change the sign of n in the original action and dial up any positive, zero, or negative value of δ_0^2 .

As before, we use the basis k_i to expand Ξ in the original action (10) and integrate over the extra dimension. Including the kink sector, the effective four dimensional action is then

$$\mathcal{S}_{\Phi+\Xi} = \int d^4x [-\varepsilon_{\phi_c} + \mathcal{L}_\phi + \mathcal{L}_\xi],$$

where the kink-only parts are given previously and the scalar Lagrangian is

$$\begin{aligned} \mathcal{L}_\xi = & \sum_{i=0}^{[b-1]} \left[\frac{1}{2} \partial^\mu \xi_i \partial_\mu \xi_i - \frac{1}{2} \delta_i^2 \xi_i^2 \right] + \int_{-\infty}^{\infty} dq \left[\frac{1}{2} \partial^\mu \xi_q \partial_\mu \xi_q - \frac{1}{2} \delta_q^2 \xi_q^2 \right] \\ & - g_{ijk}^{(3)} \phi_i \xi_j \xi_k - g_{ijkl}^{(4)} \phi_i \phi_j \xi_k \xi_l - \tau_{ijkl} \xi_i \xi_j \xi_k \xi_l. \end{aligned} \quad (12)$$

The Yukawa and self coupling factors are

$$\begin{aligned} g_{ijk}^{(3)} &= \frac{ab(b+1)}{2m} \int_{-\infty}^{\infty} dw [\phi_c \eta_i k_j k_k], \\ g_{ijkl}^{(4)} &= \frac{ab(b+1)}{4m} \int_{-\infty}^{\infty} dw [\eta_i \eta_j k_k k_l], \\ \tau_{ijkl} &= \frac{c}{4n} \int_{-\infty}^{\infty} dw [k_i k_j k_k k_l]. \end{aligned}$$

With this expanded four dimensional action, we are ready to analyse the various limits of the model with the scalar field.

B. The thin kink with a scalar field

We are considering the combined five dimensional action $\mathcal{S}_{\Phi+\Xi} = \mathcal{S}_\Phi + \mathcal{S}_\Xi$ and the various limits that arise through m , the mass of the kink. As discussed previously, we have three scenarios which are characterised by the sign of μ . In the case $\mu < 0$ there is no kink and we are left with two interacting five dimensional fields Φ and Ξ . The thick kink scenario,

$\mu = 0$, has many interacting four dimensional scalar fields, the details given by the mass spectra of ϕ_i and ξ_i and the couplings κ , g and τ . We will not dwell on these two cases, but instead concentrate our attention on the thin kink limit and explore the parameter space of b , c and n .

It was shown previously that the thin kink limit leaves only the energy density of the domain wall in the effective four dimensional action. The dynamics of all the scalar modes ϕ_i are removed and so the third and fourth Yukawa terms in (12) are eliminated. With $m \rightarrow \infty$ and b finite, the masses of all the ξ_i modes will also tend to infinity and the scalar Ξ becomes completely frozen out. To leave some remnant of Ξ in the model we have two choices: either take b to zero to counter m^2 , or choose n^2 such that it cancels $\frac{1}{2}bm^2$.

For the first choice, let n be finite and $b = \tilde{b}\Lambda^\beta$. Then $\delta_0^2 \sim \Lambda^{\beta+2\mu}$ and the mode ξ_0 has finite mass if $\beta + 2\mu \leq 0$. Since $b \rightarrow 0$ in the limit $\Lambda \rightarrow \infty$, there are in fact no bound modes, the basis k_i is not valid and we must consider Ξ as a five dimensional field. The effective action for such a limit of the parameters is³

$$\mathcal{S}_{\Phi+\Xi}^{5D} = \int d^4x [-\varepsilon_{\phi_c}] + \int d^5x \left[\frac{1}{2} \partial^M \Xi \partial_M \Xi - \frac{1}{4} b m^2 \Xi^2 - W(\Xi) \right]. \quad (13)$$

We see that the five dimensional field Ξ has nothing dynamical to couple to, and just picks up an addition to its Ξ^2 term. If we have the strict inequality $\beta + 2\mu < 0$, this addition to the mass will be zero.

The second choice which keeps some part of Ξ alive is to change the sign of n^2 and fine tune it to exactly cancel the infinite term in δ_0^2 . An exact cancellation would render the mode ξ_0 massless; we can be more general and allow a finite mass to remain by choosing

$$n^2 = -\frac{1}{2} b m^2 + n_0^2, \quad (14)$$

where n_0 is finite. Let us briefly comment on the situation where $b \rightarrow 0$, but not quickly enough to counter m (thus $-2\mu < \beta < 0$) and so we must choose n^2 as in (14). In this case there are again no bound modes and Ξ is a five dimensional field with equivalent physics as described by (13), except the quadratic term of the generated potential is replaced by $\frac{1}{2} n_0^2 \Xi^2$.

³ This result uses $\tanh^2(mw/\sqrt{2}) \rightarrow 1$ as $m \rightarrow \infty$, which ignores the fact that the distribution vanishes on a set of measure zero at the origin.

We can now restrict our analysis to the case where $\beta \geq 0$ and n^2 is of the form given by (14). As we have a non-zero b , there is at least one bound mode, and in fact only the lowest bound mode will have finite mass. The dynamics of the higher bound modes and the continuum will not be part of the model as their corresponding masses are infinite. With only ξ_0 alive, the quartic coupling terms reduce to just the one with the factor τ_{0000} . Putting all these pieces together we arrive at the four dimensional action

$$\mathcal{S}_{\Phi+\Xi}^{4D} = \int d^4x \left[-\varepsilon_{\phi_c} + \frac{1}{2} \partial^\mu \xi_0 \partial_\mu \xi_0 - W_0(\xi_0) \right]$$

with $W_0(\xi_0) = \frac{1}{2} n_0^2 \xi_0^2 + \frac{c}{4\sqrt{2\pi}} \frac{\sqrt{2} \Gamma^2(b + \frac{1}{2}) \Gamma(2b)}{\sqrt{b} \Gamma^2(b) \Gamma(2b + \frac{1}{2})} \xi_0^4.$

For large values of b the potential simplifies to

$$W_0(\xi_0) \sim \frac{1}{2} n_0^2 \xi_0^2 + \frac{c}{4\sqrt{2\pi}} \xi_0^4.$$

This analysis shows that in the thin kink limit, a five dimensional coupled scalar field is projected down to a single, localised, four dimensional scalar field ξ_0 . As the parameters n_0 and c are arbitrary, one can generate a phenomenologically suitable potential for ξ_0 , in particular the sign of n_0^2 can be changed to yield a potential which encourages a non-zero vacuum expectation value. We mention two uses of this. Most obviously this mechanism can be used to localise the standard model electroweak Higgs field to a brane, and it should be no trouble to arrange Yukawa couplings to fermion fields for mass generation. Secondly, the effective potential $W_0(\xi_0)$ has the same form as the kink potential $V(\Phi)$ and can thus support a domain wall solution, leading to the idea of nested brane worlds. Beginning with a six dimensional model with the two scalar fields Φ and Ξ and a suitable potential, one can use Φ to generate a domain wall and an effective five dimensional action and then use the lowest projected mode of Ξ in the same way to generate an effective four dimensional action. The method used does not depend on the dimensionality and one is free to generate an arbitrary number of nestings. Of course, this mechanism just deals with the particle content. The non-trivial exercise is to check that gravity can be broken down in similar stages and made to reproduce four dimensional general relativity; this will not be attempted here.

IV. ADDING A FERMION FIELD

In the previous section we performed a full analysis of the modes of a five dimensional scalar field coupled to a kink. We showed that in the thin kink limit, an effective four dimensional scalar field remains and could be potentially useful for model building. In direct analogy with this analysis we now consider a five dimensional massless fermion with Yukawa coupling to Φ , find a suitable basis for decomposition, and investigate the limiting behaviour. This is a generalisation of the well known and heavily used result that the kink supports a zero mode fermion, as first discussed in the brane context in [7]. For a partially analytic analysis of massive fermion modes confined to a thick brane in the presence of gravity, see [13].

A. Fermion modes

Fermions in five dimensions are four component spinors and their Dirac structure will be described by Γ^M with $\{\Gamma^M, \Gamma^N\} = g^{MN}$. Specifically $\Gamma^\mu = \gamma^\mu$ and $\Gamma^5 = i\gamma^5$ where $\gamma^{\mu,5}$ are the usual gamma matrices in the Dirac representation. Our action for a massless fermion coupled to the kink is $\mathcal{S}_{\Phi+\Psi} = \mathcal{S}_\Phi + \mathcal{S}_\Psi$ where

$$\mathcal{S}_\Psi = \int d^5x \left[\bar{\Psi} i \Gamma^M \partial_M \Psi - \sqrt{\frac{ad^2}{2m}} \Phi \bar{\Psi} \Psi \right]. \quad (15)$$

The kink parameters a and m are the same as before and d is a dimensionless coupling parameter. As with the scalar field Ξ , we expect the extra dimensional behaviour of Ψ to be quite different to the four dimensional part. Also, because of the Dirac structure of the fifth gamma matrix $\Gamma^5 = i\gamma^5$, we expect left- and right-handed projections of the four dimensional part to behave differently. Thus we choose the general expansion

$$\Psi(x^\mu, w) = \sum_i \psi_{Li}(x^\mu) f_{Li}(w) + \sum_i \psi_{Ri}(x^\mu) f_{Ri}(w), \quad (16)$$

where the f_{Li} and f_{Ri} are a fixed basis, the ψ_i are dynamical, $\gamma^5 \psi_{Li} = -\psi_{Li}$, $\gamma^5 \psi_{Ri} = \psi_{Ri}$ and the sum over i can be both discrete and continuous. To obtain the defining equations for the basis functions f_i , we impose that the ψ_i satisfy the massive Dirac equation by $i\not{\partial} \psi_{Li} = \sigma_i \psi_{Ri}$ and $i\not{\partial} \psi_{Ri} = \sigma_i \psi_{Li}$. Then, substituting the expansion (16) into the Euler-Lagrange equation

for Ψ we arrive at

$$\begin{aligned} & \psi_{Li} \left(-\partial_w f_{Li} + f_{Ri} \sigma_i - \sqrt{\frac{ad^2}{2m}} \phi_c f_{Li} \right) \\ & + \psi_{Ri} \left(\partial_w f_{Ri} + f_{Li} \sigma_i - \sqrt{\frac{ad^2}{2m}} \phi_c f_{Ri} \right) = 0. \end{aligned} \quad (17)$$

Since left and right Dirac components are independent and the ψ_i are arbitrary fields, both of the two parenthesised factors in equation (17) must be zero. Hence the f_{Li} and f_{Ri} must satisfy a set of two first order coupled ordinary differential equations. We turn these equations into two uncoupled second order equations

$$\begin{aligned} \left(-\frac{d^2}{dz^2} + d(d+1) \tanh^2 z - d \right) f_{Li} &= \frac{2\sigma_i^2}{m^2} f_{Li}, \\ \left(-\frac{d^2}{dz^2} + d(d-1) \tanh^2 z + d \right) f_{Ri} &= \frac{2\sigma_i^2}{m^2} f_{Ri}. \end{aligned}$$

As with the scalar field, we see that the kink sets up a symmetric modified Pöschl-Teller potential well which traps the extra dimensional component of the fermion field. This has been noted in a similar context of fat branes in [14], and in the context of a two dimensional Dirac equation in [15]. We use Appendix A of the current work to obtain the solutions; the bound modes come in pairs given by

$$\begin{aligned} (\sigma_0^d)^2 = 0 & \quad \begin{cases} f_{L0}^d(w) = F_{L0}^d \cosh^{-d} z \\ f_{R0}^d(w) = 0, \end{cases} \\ (\sigma_1^d)^2 = \frac{1}{2}(2d-1)m^2 & \quad \begin{cases} f_{L1}^d(w) = F_{L1}^d \sinh z \cosh^{-d} z \\ f_{R1}^d(w) = F_{R1}^d \cosh^{-d+1} z, \end{cases} \\ \vdots & \\ (\sigma_n^d)^2 = \frac{1}{2}(2nd - n^2)m^2 & \quad \begin{cases} f_{Ln}^d(w) = \frac{1}{\sigma_n^d} \left(\frac{dm}{\sqrt{2}} \tanh z - \frac{d}{dw} \right) f_{L,n-1}^{d-1} \\ f_{Rn}^d(w) = f_{L,n-1}^{d-1}(w). \end{cases} \end{aligned}$$

These bound state modes are valid for all positive values of d and there are $[d]$ sets of modes. For the continuum, the solutions can be found in terms of standard functions when

d is a positive integer. They are

$$\begin{aligned}
(\sigma_q^1)^2 &= \frac{1}{2}(q^2 + 1)m^2 & \begin{cases} f_{Lq}^1 = F_{Lq}^1 e^{iqz} (\tanh z - iq) \\ f_{Rq}^1 = F_{Rq}^1 e^{iqz}, \end{cases} \\
(\sigma_q^2)^2 &= \frac{1}{2}(q^2 + 4)m^2 & \begin{cases} f_{Lq}^2 = F_{Lq}^2 e^{iqz} (3 \tanh^2 z - (q^2 + 1) - 3iq \tanh z) \\ f_{Rq}^2 = F_{Rq}^2 e^{iqz} (\tanh z - iq), \end{cases} \\
&\vdots \\
(\sigma_q^d)^2 &= \frac{1}{2}(q^2 + d^2)m^2 & \begin{cases} f_{Lq}^d = \frac{1}{\sigma_n^d} \left(\frac{dm}{\sqrt{2}} \tanh z - \frac{d}{dw} \right) f_{Lq}^{d-1} \\ f_{Rq}^d = f_{Lq}^{d-1}(w). \end{cases}
\end{aligned}$$

The normalisation coefficients F can be computed using Appendix A.

Using these basis functions, we expand the original action (15) and integrate over the extra dimension. The effective four dimensional action, including the kink dynamics, is

$$\mathcal{S}_{\Phi+\Psi} = \int d^4x [-\varepsilon_{\phi_c} + \mathcal{L}_\phi + \mathcal{L}_\psi],$$

where the Lagrangian for the expanded Ψ is

$$\begin{aligned}
\mathcal{L}_\psi &= \overline{\psi_{L0}} i \not{\partial} \psi_{L0} + \sum_{i=1}^{[d-1]} \overline{\psi_i} (i \not{\partial} - \sigma_i) \psi_i + \int_{-\infty}^{\infty} dq [\overline{\psi_q} (i \not{\partial} - \sigma_q) \psi_q] \\
&\quad - h_{ijk} \phi_i \overline{\psi_{Lj}} \psi_{Rk} - h_{ikj}^* \phi_i^* \overline{\psi_{Rj}} \psi_{Lk}.
\end{aligned} \tag{18}$$

We have condensed the notation using $\psi_i = \psi_{Li} + \psi_{Ri}$ and similarly for ψ_q . For brevity in the Yukawa terms, the implicit sum over i denotes a sum over the bound modes $i = 0, 1$ and an integral over the continuum; similarly the sum over j and k denotes the sum over bound and continuum fermion modes. The effective dimensionless Yukawa coupling is

$$h_{ijk} = \sqrt{\frac{ad^2}{2m}} \int_{-\infty}^{\infty} dw [\eta_i f_{Lj}^* f_{Rk}].$$

We can compute these h . Those of importance are the couplings between the bound kink

modes and the bound fermion modes, the first few being

$$\begin{aligned}
h_{0i0} &= 0 \quad (\text{for all } i), \\
h_{001} &= \sqrt{\frac{3a}{8\sqrt{2}}} \left(\frac{d - \frac{1}{2}}{d - 1} \right)^{\frac{3}{2}} \frac{\Gamma^2(d - \frac{1}{2})}{\Gamma^2(d - 1)}, \\
h_{011} &= h_{101} = 0, \\
h_{111} &= \sqrt{\frac{3a}{8\sqrt{2}}} \frac{(d - \frac{1}{2})^{\frac{1}{2}}}{d - 1} \frac{\Gamma^2(d - \frac{1}{2})}{\Gamma^2(d - 1)}.
\end{aligned}$$

This analysis includes the well known chiral zero mode localisation when all fermion modes except ψ_{L0} are removed from (18). In this reduced model there are no Yukawa couplings between ψ_{L0} and any of the kink modes ϕ_i due to the chirality of the fermion mode.

B. The thin kink with a fermion field

With the expanded effective four dimensional action for Ψ , we can now consider the limits of the kink. With no kink ($\mu < 0$) the basis f_i is not valid and we instead obtain a model containing the coupled five dimensional fields Φ and Ψ . The thick kink scenario ($\mu = 0$) contains four dimensional interacting scalar fields ϕ_i and ψ_i . The thin kink limit ($\mu > 0$) is what we are most interested in, where the remnant of the kink sector is just the energy density. Then the Yukawa term in (15) reduces to $(dm/\sqrt{2}) \tanh^2 z \bar{\Psi}\Psi$ and we have only the parameter d left to play with. There are four scenarios, which we classify using the previous parameterisation of the limit and write $d = \tilde{d}\Lambda^\delta$.

First, if $\delta = -\mu$ then $d \rightarrow 0$, there are no bound fermion modes and the fermion is left as a five dimensional field with the action

$$\mathcal{S}_{\Phi+\Psi}^{5D} = \int d^4x [-\varepsilon_{\phi_c}] + \int d^5x \left[\bar{\Psi} i \Gamma^M \partial_M \Psi - \frac{dm}{\sqrt{2}} \theta(w) \bar{\Psi} \Psi \right],$$

where $\theta(w)$ is the step function. The fermion in this case has an unusual mass term that changes sign across the kink. Second, if $\delta < -\mu$, we have the same situation as in the first case, except the mass term disappears. Third, if $-\mu < \delta < 0$, d is not going to zero fast enough to counter m and the unusual mass term becomes infinite. The five dimensional field Ψ is thus frozen out and we are left with just the kink energy density in our effective theory.

The fourth case has $\delta \geq 0$, so $d > 0$ and there is at least one bound fermion mode. Because the masses of these modes go like m , all modes are frozen out except the zero mode ψ_{L0} . None of the Yukawa couplings h are relevant because they couple ψ_{L0} to higher fermion modes which are not dynamical. The effective action is then simply

$$\mathcal{S}_{\Phi+\Psi}^{4D} = \int d^4x \left[-\varepsilon_{\phi_c} + \overline{\psi_{L0}} i \not{\partial} \psi_{L0} \right],$$

which contains the dynamics of a single, four dimensional, chiral, massless fermion. Thus we obtain the result of Rubakov and Shaposhnikov in the thin kink limit with $d > 0$. This is of obvious importance in model building where chiral zero modes are the starting point for the particle content.

In all the limits of the models we have considered so far, the dynamical fields are either exclusively four dimensional or exclusively five dimensional. It is possible to construct an action which in the thin kink limit contains interacting four and five dimensional fields. Because the dynamics of the kink are frozen out, two extra fields are required from the outset: one which couples appropriately to the kink and provides a four dimensional zero-mode, and another which remains five dimensional. To demonstrate this idea, we take the original action for the kink and the fermion and add the scalar field Ξ with a coupling to the fermion only. The action is then

$$\mathcal{S}_{\text{all}} = \mathcal{S}_{\Phi} + \mathcal{S}_{\Psi} + \int d^5x \left[\frac{1}{2} \partial^M \Xi \partial_M \Xi - s \Xi (\overline{\Psi} \Psi^c + \overline{\Psi^c} \Psi) \right],$$

where s is a dimensionful coupling constant and the charge conjugate field is defined as $\Psi^c = \Gamma^2 \Gamma^4 \Psi^*$. We choose the kink-fermion coupling $d > 0$ such that there is at least one bound mode and follow the thin kink analysis performed above. All the massive fermion modes are frozen out and the Ξ coupling term becomes

$$s \Xi (\overline{\Psi} \Psi^c + \overline{\Psi^c} \Psi) \xrightarrow{m \rightarrow \infty} s (f_{L0})^2 \Xi (\overline{\psi_{L0}} \psi_{L0}^c + \overline{\psi_{L0}^c} \psi_{L0}). \quad (19)$$

The extra dimensional factor from the fermion zero mode becomes a delta distribution in the thin kink limit

$$(f_{L0})^2 = \frac{m \Gamma(d + \frac{1}{2})}{\sqrt{2\pi} \Gamma(d)} \cosh^{-2d} \left(\frac{mw}{\sqrt{2}} \right) \xrightarrow{m \rightarrow \infty} \delta(w).$$

The w integral can then be performed over the Yukawa term given by (19) which reduces

the action to

$$\begin{aligned} \mathcal{S}_{\text{all}}^{4\text{D}/5\text{D}} = & \int d^4x \left[-\varepsilon_{\phi_c} + \overline{\psi}_{L0} i \not{\partial} \psi_{L0} - s \Xi(w=0) (\overline{\psi}_{L0} \psi_{L0}^c + \overline{\psi}_{L0}^c \psi_{L0}) \right] \\ & + \int d^5x \left[\frac{1}{2} \partial^M \Xi \partial_M \Xi \right]. \end{aligned}$$

This is a dynamically generated model describing a four dimensional zero mode ψ_{L0} coupled at the extra dimensional point $w = 0$ to a five dimensional field Ξ . The situation can easily be reversed to have Ψ five dimensional and coupling at $w = 0$ to the four dimensional ground state mode of Ξ . Extensions to multiple four and five dimensional fields are also easily obtained.

V. CONCLUSION

Our universe may be embedded in a large extra spatial dimension and it seems sensible to look for a field theoretic description of such a scenario. We would like such a model to explain the dynamics behind the confinement of higher dimensional fields to our four dimensional subspace. We have demonstrated in this paper that the domain wall defect (the kink) is an attractive candidate for such dynamical confinement as it can localise not only massless chiral fermions but also scalars with arbitrary quartic potentials.

We have analysed in detail the kink and its modes and discussed the behaviour of these degrees of freedom in the limits of no kink, a thick kink and a thin kink. Due to the quartic potential which sets up the kink profile, the Kaluza-Klein spectrum consists of a massless bound mode, a massive bound mode and a massive continuum. In the thin kink limit all the massive modes freeze out as their mass becomes infinite. The zero mode also freezes out in this limit due to its divergent quartic self coupling. This leaves just the kink energy density in the effective four dimensional action.

A full analysis of a second scalar field coupled to the kink was performed. We showed that this field is trapped by the kink in a symmetric modified Pöschl-Teller potential well and we gave the mass spectrum for this trapping. In the thin kink limit, this extra scalar field could either freeze out completely, be a free five dimensional field, or have just its ground state mode in an effective four dimensional action. We showed that in this latter case, an arbitrary quartic potential could be generated for this mode and it could thus be used as a standard model electroweak Higgs field.

Our final analysis was that of a fermion coupled to the kink. It has been known for some time that such a model admits a massless chiral mode. We generalised this result by giving the full spectrum of fermion modes in the presence of the kink. These four dimensional fields consist of a left-handed zero mode, a certain number of pairs of left- and right-handed modes and a continuum of pairs. In the thin kink limit, we showed that the fermion can either freeze out completely, be five dimensional with or without a mass term, or reproduce the result of a localised four dimensional, massless, left-handed mode. We also demonstrated that with the kink and fermion and an extra scalar field, one can arrange things such that the left-handed zero mode couples to a five dimensional field at a single point in the extra dimension.

The results of the present work gives one the tools necessary to write down the standard model, without gauge fields, on a brane in five dimensions. With the details of the spectrum of the decomposed five dimensional fields, one can compute the phenomenology of the interactions between the massless and massive modes and also use experimental data to put an upper bound on the width of the brane. Further work would focus on the addition of gauge fields and their localisation, as well as the inclusion of gravity.

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APPENDIX A: SYMMETRIC MODIFIED PÖSCHL-TELLER POTENTIAL MODE SOLUTIONS

In this appendix we give analytic solutions to the symmetric modified Pöschl-Teller potential. The non-symmetric potential was first studied by Rosen and Morse in the context of molecular dynamics [16]. They presented solutions in terms of hypergeometric functions. Later work by Nieto [17] computed explicit forms of the bound mode solutions, including normalisation coefficients, in terms of regular functions. Rajaraman [18] gives unnormalised solutions for the bound and continuum modes for a specific case of the potential, and the hypergeometric forms of the solutions are again explored in [14]. Bound state solutions are

also expressed in terms of Gegenbauer polynomials in [15]. As far as we are aware, exact closed form solutions for the continuum modes and their normalisation factors have not been presented in the literature. We present these forms in this Appendix, along with simple expressions for the bound states, simple recurrence relations for higher modes, normalisation coefficients, and the closure relation.

The time-independent Schrödinger equation with the symmetric version of the potential takes the form

$$\left(-\frac{d^2}{dx^2} + l(l+1)\tanh^2 x - l\right)\psi_n = E_n\psi_n. \quad (\text{A.1})$$

If $l = 0$ then the solutions are just plane waves. For $l > 0$ there are a set of bound modes followed by continuum modes. The bound solutions are

$$\begin{aligned} E_0^l &= 0 & \psi_0^l(x) &= A_0^l \cosh^{-l} x, \\ E_1^l &= 2l - 1 & \psi_1^l(x) &= A_1^l \sinh x \cosh^{-l} x, \\ E_2^l &= 4l - 4 & \psi_2^l(x) &= A_2^l \left(\frac{2l-2}{2l-1} \cosh^{-l+2} x - \cosh^{-l} x \right), \\ &\vdots & & \\ E_n^l &= 2nl - n^2 & \psi_n^l(x) &= \frac{1}{\sqrt{E_n^l}} \left(l \tanh x - \frac{d}{dx} \right) \psi_{n-1}^{l-1}(x). \end{aligned}$$

The square integrable ortho-normalisation condition is

$$\int_{-\infty}^{\infty} \psi_n^l(x) \psi_{n'}^l(x) dx = \delta_{nn'}, \quad (\text{A.2})$$

and the normalisation coefficients are

$$A_0^l = \sqrt{\frac{\Gamma(l + \frac{1}{2})}{\sqrt{\pi} \Gamma(l)}}, \quad A_1^l = \sqrt{2l-2} A_0^l, \quad A_2^l = \sqrt{(2l-1)(l-2)} A_0^l.$$

These bound mode solutions are valid for all positive real values of l . There are $[l]$ bound modes² and so the mode index takes the values $n = 0, 1, \dots, [l-1]$. For the continuum we have found forms for the solutions in terms of regular functions for the case where l is a positive integer. Instead of the discrete bound mode index n , the continuum are indexed

with a continuous label $p \in \mathbb{R}$. They take the form

$$\begin{aligned} E_p^1 &= p^2 + 1 & \psi_p^1(x) &= A_p^1 e^{ipx} (\tanh x - ip), \\ E_p^2 &= p^2 + 4 & \psi_p^2(x) &= A_p^2 e^{ipx} (3 \tanh^2 x - (p^2 + 1) - 3ip \tanh x), \\ &\vdots & & \\ E_p^l &= p^2 + l^2 & \psi_p^l(x) &= \frac{1}{\sqrt{E_p^l}} \left(l \tanh x - \frac{d}{dx} \right) \psi_p^{l-1}(x). \end{aligned}$$

The delta distribution ortho-normalisation condition is

$$\int_{-\infty}^{\infty} \psi_p^l(x) \psi_{p'}^l(x) dx = \delta(p - p'), \quad (\text{A.3})$$

and the normalisation coefficients are

$$A_p^1 = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{p^2 + 1}}, \quad A_p^2 = \frac{1}{\sqrt{p^2 + 4}} A_p^1.$$

These continuum modes are valid only for $l = 1, 2, 3, \dots$; for other values of l one must resort to the hypergeometric form. Note that the bound modes are orthogonal to the continuum modes. Finally, we state the closure relation

$$\sum_{n=0}^{[l-1]} \psi_n^l(x) \psi_n^l(x') + \int_{-\infty}^{\infty} \psi_p^l(x) \psi_p^l(x') dp = \delta(x - x').$$

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